Adiabatic Theory and Wave Action Density

Wave Adiabatic Theory / Wave Kinetics

- frequently encounter continuum problems with slowly varying parameters $\Rightarrow$ adiabatic theory needed

$\Rightarrow$

- $\boxed{\text{wave kinetic equation}}$ (consequence of Liouville Thm.)

$$\partial_t N + (\underline{V}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \partial_{\underline{k}} N$$

$$= \cancel{C(N)}$$; obvious analogy to Boltzmann Eqn.

$N \equiv \mathcal{E} / \omega_{\underline{k}}$ $\equiv$ wave action density / wave quanta density

wave energy density $\mathcal{E} = \frac{\partial}{\partial \omega} (\omega \epsilon_{\underline{k}}) \Big|_{\omega_{\underline{k}}} \frac{|E_{\underline{k}}|^2}{8\pi}$, for e.s. waves

$N \Leftrightarrow \mathcal{E}$

characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} \hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V})$$

refraction by shear

refraction by parametric variation

- need:

$$\omega < \frac{1}{\lambda} \frac{d\lambda}{dt}$$ $\lambda \equiv$ parameter

$\rightsquigarrow$ $\boxed{\text{space and time scale separation}}$

$$\frac{1}{N} (\underline{V}_{gr} \cdot \nabla N) \ll \omega \Rightarrow \underline{\Sigma} \cdot V_{gr} < \omega$$

$\tilde{}$(N) → interactions with comparable scale.
ignore here.

Examples :

- linear theory of Langmuir turbulence
  i.e. when will phonon grow ?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasmon
  energy → net impact ?!
- drift waves and sheared flow.
→ transport equation, super-fluids $\quad N = \frac{\mathcal{E}}{\omega}$

→ dynamics?!

Fundamentals of wave kinetics

→ where does conservation of action emerge from ?

→ answer: | phase symmetry of wave train underlies wave kinetics |

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

Transport Eqn - @M

$$\frac{\partial n}{\partial t} + v_{gr} \cdot \nabla n - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} n = C(n)$$

$$\frac{\partial n}{\partial t} + \frac{\partial \hbar\omega}{\partial \hbar \underline{k}} \cdot \nabla n - \frac{\partial \hbar\omega}{\partial x} \cdot \nabla_{\hbar k} n = C(n)$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial G}{\partial \underline{p}} \cdot \nabla n - \frac{\partial G}{\partial x} \cdot \nabla_{p} n = C(n)}$$

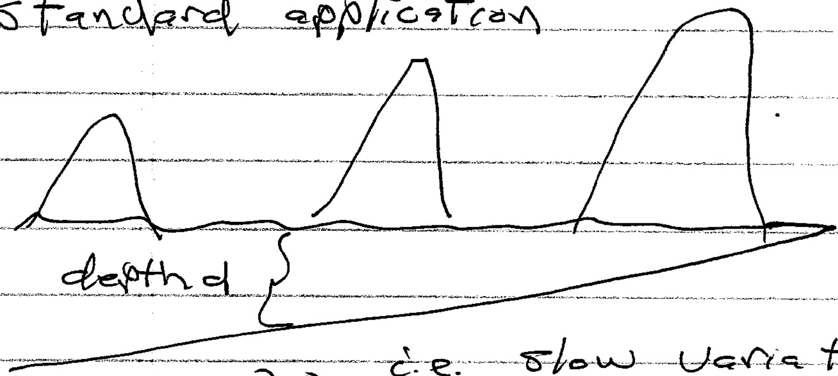
valid for:

$$\frac{1}{\epsilon}\frac{\partial G}{\partial x} < 1/\lambda_{DB}$$

$$\lambda_{DB} = \hbar/p.$$

→ standard application

⇒



beach ↔

{ waves in shallow water

→ i.e. slow variation

$$\frac{1}{d}\frac{d}{dx}d(x) \ll k$$

- in flux of wave energy

- depth $H(x,y)$ decreases

⇒ wave amplification, breaking.

Derivation

Consider a system, [like ideal fluid MHD] acoustics which can be described in terms of displacement $\underline{\xi}$:

i.e. $$\underline{\xi} = \mathrm{re}\{A\, e^{i\overset{\text{phase}}{\phi}} + A^* e^{-i\phi}\}$$

displacement can be viewed as excitation level

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x}\, \mathcal{L}(\xi)$$

- Envision a wave train, with slowly varying amplitude, so eikonal approach optimal i.e. fast variation in phase, a la WKB:

$$S = \int dt \int d\underline{x}\, \mathcal{L}(\omega, \underline{k}, \underset{\text{amplitude}}{a})$$

$$\underline{k} = \underline{\nabla}\phi$$
$$\omega = -\partial_t \phi$$

$$= \int dt \int d\underline{x}\, \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

- neglect all corrections to eikonal theory.

- here L corresponds to period-averaged Lagrangian
- ϕ undetermined to const $\rightarrow$ phase symmetry!

$\therefore$ to vary:

$$\delta S/\delta a = 0$$

$$\delta S/\delta \phi = 0$$

Now, in linear theory:

$$\left[G(k,\omega) \Leftrightarrow G \right]$$

- $$\mathcal{L} = G(\omega, k)\, a^2$$

continuum,

$G(\omega, \underline{k}) = 0$ dispn

$\omega^2 = k^2 c_s^2$

i.e. for MHD, as in wave section:

$$\mathcal{L} = \tfrac{1}{2}\rho \dot{\underline{\xi}}^2 - \tfrac{1}{2}\rho \left[\underline{\underline{D}}(k, x, t)\right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

$\rightarrow$ eikonal form of stiffness matrix ($\rightarrow$ potential energy)

$$\Rightarrow \underline{\xi} \cdot \underline{\underline{M}} \cdot \underline{\xi}$$

$M(k, \omega, \theta)$, as for linear waves

if: $\underline{\xi} = \underline{A} e^{i\phi} + \underline{A}^* e^{-i\phi}$

$$\underline{\underline{i}} \quad G(\omega, \underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2\right]$$

Now, 1) $\delta S / \delta a = 0$

$$\Rightarrow \quad G(\omega, k) = 0 \qquad \rightarrow \text{dispn relation}$$

but

$$G(\omega, k) = \rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2$$

$$= \rho\omega^2 - D^2$$

$\hookrightarrow$ stiffness fctn.

$\Rightarrow$ dispn. relation

2) $\delta S / \delta \phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)} \delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})} \delta(\phi_{\underline{x}}) \right\}$$

end pts fixed, i b p

$$= \int dx \int d^3x \left\{ \partial_t\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial(\underline{\nabla}\phi)}\right) \right\} \delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

Now, have: $\quad G(\underline{k}, \omega) = 0 \qquad$ (disp. reln.)

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}}\cdot d\underline{k} = 0$$

$$\therefore \quad v_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega} \qquad \text{(along } \omega\text{)}$$

$\therefore$

$$\partial_t\left((\partial G/\partial\omega)\, a^2\right) + \nabla\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\,\frac{\partial G}{\partial \omega}\, a^2\right] = 0$$

and so $\quad N \equiv \dfrac{\partial G}{\partial \omega}\, a^2$

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\underline{v}_{gr} N\right) = 0$$

(N not $\underline{\underline{\text{yet}}}$ action)

Also note energy is conserved $\Leftrightarrow$ G invariant to time translations.

so, Noethers thm $\Rightarrow$ there exists an [redacted] energy conservation equation

have $\quad \mathcal{L} = G(k,\omega)\, a^2$

$$\partial \mathcal{L}/\partial a = 0 \Rightarrow G(\omega, k) = 0$$

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

and of course:

$$\nabla \times \underline{k} = 0, \quad \text{as } \underline{k} = \nabla\phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}, \quad \text{as } \partial_t \nabla\phi = -\nabla\left(-\frac{\partial}{\partial t}\phi\right) \checkmark$$

Now, $\mathcal{L} = 0$, as $G(\underline{k},\omega) = 0$

as expect $\quad \frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow N, \qquad \omega \frac{\partial \mathcal{L}}{\partial \omega} \Rightarrow \mathcal{E}$

$\Rightarrow 0$, [redacted] creatively $\to$ dispn relation satisfied

$$\frac{\partial}{\partial t}\left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - [redacted]\, \mathcal{L}\right) + \nabla\cdot\left[-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right] = 0$$

$$-\frac{\partial G/\partial k}{\partial G/\partial \omega}\, \frac{\partial G}{\partial \omega} a^2$$

[redacted] $v_{gr}\, \mathcal{E}$

$$\partial_t \left(\omega \mathcal{L}_\omega - \mathcal{L} \right) + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{u}} \right) = 0$$

check:

$$(\partial_t \omega) \mathcal{L}_\omega + \omega \, \partial_t (\mathcal{L}_\omega) - \partial \mathcal{L} / \partial t$$
$$+ \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{u}} \right) = 0$$

$\Rightarrow$

but $\quad \partial_t \mathcal{L}_\omega = \underline{\nabla} \cdot (\mathcal{L}_{\underline{u}})$

$$\therefore \; (\mathcal{L}_\omega)(\partial_t \omega) + \cancel{\omega \, \nabla \cdot (\mathcal{L}_u)} - \cancel{\omega \, (\underline{\nabla} \cdot \mathcal{L}_{\underline{u}})}$$
$$- \left(\frac{\partial \mathcal{L}}{\partial \underline{u}} \right) \cdot \underline{\nabla} \omega - \frac{\partial \mathcal{L}}{\partial t}$$

but $\quad \partial_t \underline{u} = - \underline{\nabla} \omega \qquad$ (order derivs)

$$(\partial_t \omega)(\mathcal{L}_\omega) + (\partial_t \underline{u}) \cdot \frac{\partial \mathcal{L}}{\partial \underline{u}} - \frac{\partial \mathcal{L}}{\partial t} = 0 \quad \checkmark$$

(identity)

$$\Rightarrow \boxed{\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right\} + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{u}} \right) = 0}$$

But $\quad G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Poynting Form

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\partial \omega} \rightarrow \left\{ \begin{array}{l} \text{wave} \\ \text{energy density} \end{array} \right.$$

so

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega \rightarrow \left\{ \begin{array}{l} \text{wave} \\ \text{action density !} \end{array} \right.$$

$$= N(\underline{k}, \underline{x}, t)$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \nabla N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial\omega/\partial\underline{x}\, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$
(i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{v}_{gr} N \right] = 0$$

Observe:

$\rightarrow$ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

$\rightarrow$ continuity-type equation in $\underline{x}$-space, for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (\underline{v}_{gr} N) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

Now $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, following packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr} \cdot \underline{\nabla} k$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \frac{\partial\omega}{\partial\underline{k}} \cdot \frac{\partial\underline{k}}{\partial\underline{x}}$$

$\therefore$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial D}{\partial \underline{x}}$$ agree!

→ Now, can convert from N to $\mathcal{E}$!

i.e. $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{rays} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\bigg|_{\text{rays}} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\bigg|_{\text{rays}} = 0$$

Now $$\frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial \underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial \underline{k}}\cdot\frac{d\underline{k}}{dt}$$

From eikonal eqns:

$$= \partial_t\omega + \frac{\partial\omega}{\partial \underline{x}}\cdot\cancel{\frac{\partial\omega}{\partial \underline{k}}} - \frac{\partial\omega}{\partial \underline{k}}\cdot\cancel{\frac{\partial\omega}{\partial \underline{x}}}$$

so <u>if</u> $\partial_t\omega = 0$

$\therefore$ $$\frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0.$$

<u>so</u> $$\partial_t\mathcal{E} + \underline{v}_{gr}\cdot\underline{\nabla}\,\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_{k}\,\mathcal{E} = 0$$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{v}_{gr}\,\mathcal{E}\right] = 0}$$

conserved energy density

so, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot [V_{gr} \mathcal{E}] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

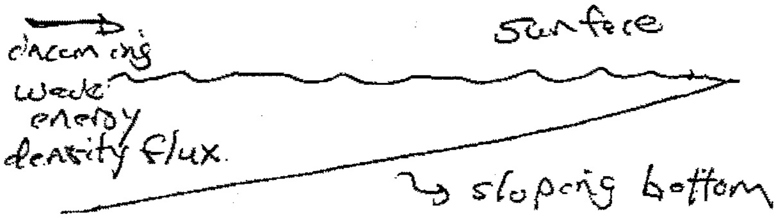
$$\Rightarrow \boxed{\underline{\nabla} \cdot [\underset{\rightarrow}{V_{gr}} \mathcal{E}] = 0}$$

incompressible wave energy flux !

$\Rightarrow$ V_{gr} drops $\Rightarrow$ $\mathcal{E} \uparrow$ $\Rightarrow$ blocking, breaking

③ The beach....

Consider:



$H = H(x)$

Now, in shallow water
$(\lambda > H)$

$$\frac{\partial h}{\partial t} + \frac{\partial (vh)}{\partial x} = 0$$

slope
↓

shallow
water eqns.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

$$v = \cancel{v_0} + \tilde{v} \quad , \quad h = H + \tilde{h}$$

$$\Rightarrow \quad -i\omega \tilde{h} + ikH\tilde{v} = 0$$
$$-i\omega \tilde{v} = -ikg\tilde{h}$$

$$\therefore \Rightarrow \omega^2 = k^2 gH \quad \text{is dispersion relation}$$

$\Rightarrow$ analogy with acoustics is obvious

$h \Leftrightarrow \rho \qquad c_s^2 = gH$

$v \Leftrightarrow v \qquad$ etc.

energy

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$\Rightarrow$ $(1) \times \tilde{v} + (2) \times \left(g \frac{\tilde{h}}{H} \right)$

$$\therefore \quad \frac{\partial}{\partial t} \tilde{v}^2 = -g \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial}{\partial t} \tilde{h}^2 = -g \frac{H}{\cancel{H}} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial t} \left(\frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H} \right) + \frac{\partial}{\partial x} \left(g \tilde{h} \tilde{v} \right) = 0$$

is energy theorom

$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g \tilde{h}^2}{2H}$ is wave energy density

$\omega / k = (gH)^{1/2}$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(v_{gr} \mathcal{E} \right) = 0$$

$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty\,\mathcal{E}_\infty = \underline{I}$

$\underline{I}$ ↓ incoming wave flux

$V_g = \sqrt{gH(x)}$ ↳ shallow water waves have zero dispersion

$\therefore \quad \sqrt{g(H(x))}\;\mathcal{E}(x) = \underline{I}$

as $x \to$ shore $V_g \downarrow$ so wave energy ~~[illegible]~~ must increase.

Now $\mathcal{E}(x) = \dfrac{\overline{\tilde{v}^2}}{2} + \dfrac{g\overline{\tilde{h}^2}}{2H} \simeq \dfrac{g\tilde{h}^2}{\cancel{2}H}$

$$\sqrt{gH(x)}\;\frac{g\tilde{h}^2}{\cancel{2}H(x)} = I$$

$$\frac{\tilde{h}^2}{H(x)^2} = \frac{\cancel{2}I}{(\sqrt{g})^3}\left(\sqrt{H(x)}\right)^{-3}$$

i.e. $\boxed{(\tilde{h}/H)^2 \sim (\text{const})\, I \big/ (H(x))^{3/2}}$

i.e. $\tilde{h}/H \to 1 \Leftrightarrow$ breaking $\Leftarrow$ as $H(x)$ drops.

N.B.:

→ if know bottom profile, can deduce displacement profile, and approximate breaking point.

→ 2D bottom contours ⇒ wave refraction

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial \underline{x}} = -kg\left(\frac{\partial}{\partial \underline{x}} H(x,y)\right)$$

i.e. wave fronts tend to align with bottom contours approaching shore.